MATH 254

Trial Test #1

To help you prepare for Exam #1, Thursday, February 17th. You may work in groups. This trial test will take approximately 5+ hours.

NSS: Chap 1.1-1.5; 2.1-2.3; 3.1-3.4. Lecture Notes L 1-6, L 7-10 pgs. 1-11.

1. Skills:

(a) $\dfrac{\ln 125}{\ln 5} =$

(b) $\ln 9 + \ln 27 = a \ln 3$. What is a?

(c) $\left.\dfrac{d}{dx} x \ln(1+x^2)\right|_{x=1} =$

(d) $\dfrac{d}{dx} \exp \displaystyle\int_a^x p(s)ds =$ where a is constant and $p(s)$ is continuous. $a \le s \le x$

(e) $\displaystyle\int_0^2 \frac{dx}{4+x^2} =$

(f) Draw the graphs of (i) $f(y) = y^2(1-y)(2-y)$,
(ii) $f(y) = y \ln y(2-y)$

(g) $\dfrac{1}{x^3 - 6x^2 + 11x - 6} = \dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$. What are A, B, C, a, b, c?

(h) Given the curve $y(1-y) - x = 0$ in the x, y plane, draw the graph of $\begin{cases}\text{both} \\ \text{all}\end{cases}$ solution branches.

Over what range of x are $y\pm(x)$ defined? Calculate $\dfrac{dy}{dx}$ at $x = \frac{1}{8}$ where $y+(x)$ is the larger of the 2 roots.

Methods:

	Equation	Order	L or NL	A or NA
1.	$\frac{d^3y}{dx^3} + y = 7x$			
2.	$\frac{dy}{dx} = 4y$			
3.	$\frac{dy}{dx} = 4y + 6$			
4.	$\frac{dy}{dx} = 4y + 6x$			
5.	$\frac{dy}{dx} = 4y^2 + 4$			
6.	$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}$			
7.	$\frac{d^2y}{dx^2} + \frac{1}{2}y^2 - 4y = 0$			
8.	$\frac{d^2y}{dx^2} - xy = 0$			
9.	$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = \cos 3t$			
10.	$\frac{d^4y}{dt^4} = t^2$			
11.	$\frac{d^2y}{dt^2} + \sin y = 0$			
12.	$\frac{d^2x}{dt^2} + x - x^3 = 0$			

3. Find the general solutions for #'s (2), (3), (4), (5) of Q.2.
Then find the particular solution which satisfies the initial conditions.

Q.2, #2 $y(1) = 4$

Q.2, #3 $y(0) = 1$

Q.2, #4 $y(0) = 1$

Q.2, #5 $y(0) = 0$

4. Solve the following initial value problems.

$$\frac{dy}{dx} = y(3-y), \qquad y(0) = 2$$

$$\frac{dy}{dx} = y(3-y), \qquad y(0) = 4$$

$$\frac{dy}{dx} = y(3-y), \qquad y(0) = -1$$

$$\frac{dy}{dx} = y(1-y) - \frac{5}{36}, \qquad y(0) = \frac{1}{2}$$

$$\frac{dy}{dx} = y(1-y) - \frac{5}{36}, \qquad y(0) = \frac{1}{12}$$

$$\frac{dy}{dx} = y(1-y) - \frac{5}{36}, \qquad y(0) = 1$$

5. For each case in Q.4, use the graphical method for first order autonomous equations to verify your answer qualitatively.

6. Find implicit solutions for

a. $(1+y^2)\dfrac{dy}{dx} + y\tan x = 0$

b. $\dfrac{dy}{dx} = -\dfrac{\sin x}{y}$

c. $\dfrac{dy}{dx} = \dfrac{-x+x^3}{y}$

and in particular find the members of the solution families which pass through the point

(a) $x = 0,\ y = 1$

(b) $x = 0,\ y = 2$

(c) $x = \sqrt{2},\ y = 0$

Applications:

7. At time zero, a 200 gallon tank initially containing 50 gallons of pure water is filled with a brine solution containing $\frac{1}{2}$ lb. salt per gallon at a rate of 4 gals/min. It is drained at the rate of 3 gals./min. Find $x(t)$, the # of lbs. of salt after t mins, and $C(t) = \frac{x(t)}{V(t)}$, the concentration in lbs./gal. where $V(t)$ is the volume of the mixture at time t minutes. What is $x(0), C(0), C(50), C(100), C(150)$? At what time t_1 does the tank overflow? Draw $C(t)$ as a solid line for $0 \le t < t_1$ and as a dotted line for $t > t_1$.

8. A dead body is found at 3 am. At the time of discovery (t_d hours after the time of death), its temperature was 88.6°F. An hour later, it had dropped to 83.6°F. If the ambient temperature remained at a constant 58.6°F, calculate the time of death.

9. A body of mass $M = 10$ kg. falls from rest under the joint influences of gravity and friction according to the law

$$M\frac{dv}{dt} = -Mg + kv^2$$

where $v(t)$ is its velocity measured positive upwards (assume $g = 10$ m/sec^2 and k is 0.01 kg/meter). Solve for $v(t)$. What is $\lim_{t\to\infty} v(t)$?

10. Consider, for $y \geq 0$,

$$\frac{dy}{dt} = -y \ln y$$

What are the equilibrium solutions? Are they stable (Lyapunov or asymptotic) or unstable? What is the basin of attraction for the asymptotically stable solution? Find the general solution.

11. Find the exact solution to

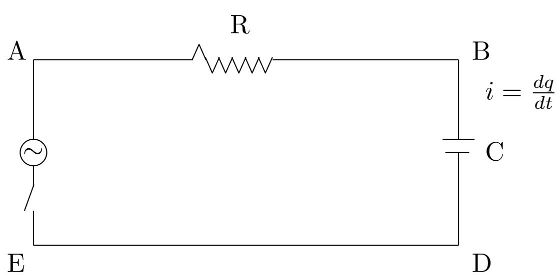
$$\frac{dy}{dt} = y(1-y)(2-y), \qquad y(0) = 3/2.$$

12. Use `slopes` to draw solution curves through several appropriately chosen points and the direction fields for

$$\text{a.} \quad \frac{dy}{dx} = \frac{-y(1-x)}{x(1-y)} \qquad\qquad \text{b.} \quad \frac{dy}{dx} = \frac{-\sin x - y}{y}$$

For (a), take the ranges of x and y to be $0 < x < 3, 0 < y < 3$ and the chosen points to be $(1,2), (\frac{1}{2}, \frac{1}{2}), (2,2), (2,1)$. For (b), Take the ranges of x and y to be $-4 < x < 4, -4 < y < 4$ and the points to be $(\pi, 1), (0,2), (0,1), (0,3)$. Draw the solution curve for (a) through $(\frac{1}{2}, \frac{1}{2})$ and for (b) through $(0,3)$.

13. Consider the RC circuit



where when the switch is closed $V_A - V_E = V(t)$.

For (a) $V(t) = V_0 \qquad t \geq 0$ and $q(0) = 0$

(b) $V(t) = V_0 \qquad 0 \leq t < T$

$\qquad 0 \qquad t > T$

$\qquad q(0) = 0$

(c) $V(t) = V_0 \cos \omega t \qquad q(0) = 0$

find $q(t)$

Hint: $\displaystyle\int^t e^{at} \cos \omega t = \frac{e^{at}}{a^2 + \omega^2}(a \cos \omega t + \omega \sin \omega t)$

Can you show this?

For the derivation of the equations for L, R, C circuits, see Lecture Notes 7-10, pgs. 2-3.